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A general method for deriving Bäcklund transformations for the Ernst equation

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Abstract. A general method for deriving Bäcklund transformations for the Ernst equation is presented. It is based on an ansatz of Clairin's type and the consistency conditions of the resulting system of differential equations. It is found that a simple ansatz gives Ehler's transformation, even though the functions appearing in the ansatz belong to a wide class of functions.

The use of Bäcklund transformations to find new solutions of non-linear partial differential equations has been extended recently to a number of such equations, some of which are the basic equations of a variety of physical problems (Miura 1976). One such equation is the Ernst equation (Ernst 1968), which is the fundamental equation for vacuum stationary axially symmetric spacetimes, and also occurs in Yang-Mills models (Witten 1979, Forgacs *et al* 1980) and the non-linear σ model (Sanchez 1982). A limited number of such transformations has been found (Kramer *et al* 1980, Cosgrove 1979, 1981). Recently attempts have been made to obtain Bäcklund transformations for this equation in a unified way (Omote and Wadati 1981, Harrison 1983), which were successful.

In the present paper a systematic method for finding Bäcklund transformations is presented, which is based on Clairin's approach to these transformations (Lamb 1974, Omote and Wadati 1981) and the consistency conditions of the resulting partial differential equations (Forsyth 1959), whose number is greater than the number of their dependent variables. First the simplest ansatz of Clairin's type is made, which leads to a system of four equations. The consistency conditions of these equations add one more equation. The resulting system of the five equations is solved. Subsequently a more general ansatz is considered, which again leads to a system of four equations. One more equation is obtained from the consistency conditions of these equations. The five equations reduce to a system of two first-order partial differential equations for one dependent variable and the general solution of each of the resulting two equations is found. Then the common solution of the two equations is obtained in the form of an arbitrary function of certain arguments. For each choice of the arbitrary function we obtain a Bäcklund transformation. A pseudopotential is introduced in a 'natural' way. The equations which the pseudopotential satisfies are found and then these equations are solved. Substituting the expression for the pseudopotential in the Bäcklund transformations and solving them we find that the new solution is related to the old by an Ehler transformation (Ehler 1957). This result holds for any choice of the above-mentioned arbitrary function. Also Ehler's transformation is found without solving the equations for the pseudopotential and without solving the equations which express the Bäcklund transformations.

The method described in this work can be used to derive Bäcklund transformations not only for the Ernst equation but also for other equations.

Let x, y, z be cartesian coordinates and let $\rho=(x^2+y^2)^{1/2}$. Then if we put

$$\xi=\rho+\mathrm{i}z \qquad \zeta=\rho-\mathrm{i}z \tag{1}$$

the Ernst equation for axially symmetric gravitational fields written in terms of the complex Ernst potential $E=E(\rho, z)=E(\xi, \zeta)$ takes in the variables ξ and ζ the form

$$\partial_\xi\partial_\zeta E+\frac{1}{4\rho}(\partial_\xi E+\partial_\zeta E)+\frac{2}{E+\bar{E}}(\partial_\xi E)(\partial_\zeta E)=0. \tag{2}$$

We shall try to find Bäcklund transformations following Clairin's method (Lamb 1974, Omote and Wadati 1981). Let E and E' be two solutions of (2) and let us assume that

$$\partial_\xi E'=\gamma\partial_\xi E \qquad \partial_\zeta E'=\delta\partial_\zeta E \tag{3}$$

where $\gamma=\gamma(E, \bar{E}, E', \bar{E}')$, $\delta=\delta(E, \bar{E}, E', \bar{E}')$ and $\partial_\alpha=\partial/\partial_\alpha$, $\alpha=\xi, \zeta$. Then if we define $\nabla_\gamma, \bar{\nabla}_\gamma, \nabla_\delta, \bar{\nabla}_\delta$ by the relations

$$\begin{aligned} \nabla_\gamma &= \partial_E+\gamma\partial_{E'} & \bar{\nabla}_\gamma &= \partial_{\bar{E}}+\bar{\gamma}\partial_{\bar{E}'} \\ \nabla_\delta &= \partial_E+\delta\partial_{E'} & \bar{\nabla}_\delta &= \partial_{\bar{E}}+\bar{\delta}\partial_{\bar{E}'} \end{aligned} \tag{4}$$

we obtain, since E and E' are solutions of (2),

$$\nabla_\delta\gamma+\frac{2\gamma}{E+\bar{E}}-\frac{2\gamma\delta}{E'+\bar{E}'}=0 \qquad \bar{\nabla}_\gamma\gamma=0 \qquad \gamma-\delta=0. \tag{5}$$

Also if the above equations hold the integrability condition $\partial_\zeta\partial_\xi E'=\partial_\xi\partial_\zeta E'$ of (3) is satisfied. Therefore we must have

$$\gamma=\delta \tag{6}$$

where γ is the common solution of equations

$$\partial_E\gamma+\gamma\partial_{E'}\gamma+\frac{2\gamma}{E+\bar{E}}-\frac{2\gamma^2}{E'+\bar{E}'}=0 \tag{7}$$

$$\partial_{\bar{E}}\gamma+\bar{\gamma}\partial_{\bar{E}'}\gamma=0. \tag{8}$$

We shall try to find solutions of (7) and (8) of the form

$$\gamma=z_1(E+\bar{E}, E'+\bar{E}')+\mathrm{i}z_2(E+\bar{E}, E'+\bar{E}'). \tag{9}$$

Then if we put

$$x_1=\tfrac{1}{2}(E+\bar{E}) \qquad x_2=\tfrac{1}{2}(E'+\bar{E}') \tag{10}$$

$$p_{ij}=\partial z_i/\partial x_j \qquad i,j=1,2 \tag{11}$$

we obtain from (7) and (8) in real variables a system of four equations. If we solve this system for p_{ij} we find

$$\begin{aligned} p_{11}&=-\frac{z_1^2+z_2^2}{x_2} & p_{12}&=-\frac{1}{x_1}+\frac{2z_1}{x_2} \\ p_{21}&=-\frac{z_1^2+z_2^2}{z_2x_1}+\frac{z_1(z_1^2+z_2^2)}{z_2x_2} & p_{22}&=\frac{z_1}{z_2x_1}-\frac{z_1^2-z_2^2}{z_2x_2}. \end{aligned} \tag{12}$$

The conditions of consistency of a system of simultaneous partial differential equations of the first order, if the number of equations is an exact multiple of the number of dependent variables involved, is given by Forsyth (1959). To write these conditions let p_{ij} be defined by (11), where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and let the number of equations be rm, where of course $r \leq n$. Let us take the rm equations of the system and solve them with respect to p_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, r$. The solution will be of the form

$$p_{ij} = \partial z_i/\partial x_j = f_{ij}(z_1, \ldots, z_m, x_1, \ldots, x_n, p_{\lambda,r+1}, \ldots, p_{\lambda,n}) \qquad \lambda = 1, \ldots, m. \tag{13}$$

Then it can be shown that for consistency of the system of equations the following conditions must be satisfied (Forsyth 1959):

$$\begin{aligned}\frac{\partial f_{ij}}{\partial x_a} - \frac{\partial f_{ia}}{\partial x_j} + \sum_{\lambda=1}^{m}\left(f_{\lambda a}\frac{\partial f_{ij}}{\partial z_\lambda} - f_{\lambda j}\frac{\partial f_{ia}}{\partial z_\lambda}\right)& \\ + \sum_{s=1}^{m}\sum_{\mu=r+1}^{n}\left(\frac{\partial f_{ij}}{\partial p_{s\mu}}\frac{\partial f_{sa}}{\partial x_\mu} - \frac{\partial f_{ia}}{\partial p_{s\mu}}\frac{\partial f_{sj}}{\partial x_\mu}\right)& \\ + \sum_{s=1}^{m}\sum_{\mu=r+1}^{n}\sum_{\lambda=1}^{m}\left[\left(\frac{\partial f_{ij}}{\partial p_{s\mu}}\frac{\partial f_{sa}}{\partial z_\lambda} - \frac{\partial f_{ia}}{\partial p_{s\mu}}\frac{\partial f_{sj}}{\partial z_\lambda}\right)p_{\lambda\mu}\right] &= 0\end{aligned} \tag{14}$$

where $i = 1, \ldots, m$, $a = j+1, \ldots, r$, $j = 1, \ldots, r-1$, and

$$\sum_{s=1}^{m}\left(\frac{\partial f_{ij}}{\partial p_{s\mu}}\frac{\partial f_{sa}}{\partial p_{l\tau}} - \frac{\partial f_{ia}}{\partial p_{s\mu}}\frac{\partial f_{sj}}{\partial p_{l\tau}} + \frac{\partial f_{ij}}{\partial p_{s\tau}}\frac{\partial f_{sa}}{\partial p_{l\mu}} - \frac{\partial f_{ia}}{\partial p_{s\tau}}\frac{\partial f_{sj}}{\partial p_{l\mu}}\right) = 0 \tag{15}$$

where i, $l = 1, \ldots, m$, $a = j+1, \ldots, r$, μ, $\tau = r+1, \ldots, n$, $j = 1, \ldots, r-1$.

If $m = 1$ we have one dependent variable, which we call z, and r equations. Let $p_j = \partial z/\partial x_j = f_j(z, x_1, \ldots, x_n, p_{r+1}, \ldots, p_n)$. In this case (15) are satisfied while (14) become

$$\frac{\mathrm{d}f_j}{\mathrm{d}x_a} - \frac{\mathrm{d}f_a}{\mathrm{d}x_j} + \sum_{\mu=r+1}^{n}\left(\frac{\partial f_j}{\partial p_\mu}\frac{\mathrm{d}f_a}{\mathrm{d}x_\mu} - \frac{\partial f_a}{\partial p_\mu}\frac{\mathrm{d}f_j}{\mathrm{d}x_\mu}\right) = \sum_{s=1}^{n}\left(\frac{\mathrm{d}F_a}{\mathrm{d}x_s}\frac{\partial F_j}{\partial p_s} - \frac{\mathrm{d}F_j}{\mathrm{d}x_s}\frac{\partial F_a}{\partial p_s}\right) \equiv [F_a, F_j] = 0 \tag{16}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}x_s} = \frac{\partial}{\partial x_s} + p_s\frac{\partial}{\partial z} \tag{17}$$

and

$$F_j = p_j - f_j(z, x_1, \ldots, x_n, p_{r+1}, \ldots, p_n).$$

Generally suppose that we have a set of equations

$$F_1 = 0,\ F_2 = 0, \ldots, F_m = 0 \tag{18}$$

where $F_1, F_2, \ldots, F_m$ are regular functions of z, $x_1, \ldots, x_n$, $p_1, \ldots, p_n$. Then a necessary and sufficient condition for the set of equations to be consistent is that

$$[F_a, F_j] = 0 \tag{19}$$

for all combinations of a and j (Forsyth 1959).

The expressions (12) do not have p_{ij} on their right-hand side. Therefore they satisfy the conditions (15). They must also satisfy the conditions (14). More specifically this must happen for $i=j=1$, $a=2$ and for $i=2$, $j=1$, $a=2$. This happens in both cases if

$$z_1^2+z_2^2=x_2^2/x_1^2. \tag{20}$$

The only common solution of the system of five equations (12) and (20) is

$$\gamma=z_1+\mathrm{i}z_2=(x_2/x_1)[1+cx_1x_2+\mathrm{i}(-2cx_1x_2-c^2x_1^2x_2^2)^{1/2}] \tag{21}$$

where c is an arbitrary real constant, for which $-2cx_1x_2-c^2x_1^2x_2^2>0$. It is obvious that the above expression (Omote and Wadati 1981) is the unique γ of the form of (9).

To study a more general case let us write

$$y_1=\tfrac{1}{2}(E+\bar{E}) \qquad y_2=\tfrac{1}{2}\mathrm{i}(E-\bar{E}) \qquad y_3=\tfrac{1}{2}(E'+\bar{E}') \qquad y_4=\tfrac{1}{2}\mathrm{i}(E'-\bar{E}') \tag{22}$$

and let us assume that

$$\gamma=z_1(y_1,y_2,y_3,y_4)+\mathrm{i}z_2(y_1,y_2,y_3,y_4). \tag{23}$$

Then if we substitute the expression (23) in (7) and (8), define p_{ij} as in (11) but with $i=1,2$, $j=1,2,3,4$, and solve the resulting four equations for p_{kl}, k, $l=1,2$, we obtain

$$\begin{aligned}
p_{11}&=-z_1p_{13}+z_2p_{14}-\frac{z_1}{y_1}+\frac{z_1^2-z_2^2}{y_3}\\
p_{12}&=-z_1p_{14}-z_2p_{13}-\frac{z_2}{y_1}+\frac{2z_1z_2}{y_3}\\
p_{21}&=-z_1p_{23}+z_2p_{24}-\frac{z_2}{y_1}+\frac{2z_1z_2}{y_3}\\
p_{22}&=-z_1p_{24}-z_2p_{23}+\frac{z_1}{y_1}-\frac{z_1^2-z_2^2}{y_3}.
\end{aligned} \tag{24}$$

If we write

$$z_i=(y_3/y_1)w_i \tag{25}$$

$$q_{ij}=\partial w_i/\partial y_j \tag{26}$$

the system (24) becomes

$$\begin{aligned}
q_{11}&=\frac{y_3}{y_1}(-w_1q_{13}+w_2q_{14})-\frac{1}{y_1}w_2^2\\
q_{12}&=-\frac{y_3}{y_1}(w_1q_{14}+w_2q_{13})+\frac{1}{y_1}w_2(w_1-1)\\
q_{21}&=\frac{y_3}{y_1}(-w_1q_{23}+w_2q_{24})+\frac{1}{y_1}w_1w_2\\
q_{22}&=-\frac{y_3}{y_1}(w_1q_{24}+w_2q_{23})-\frac{1}{y_1}w_1(w_1-1).
\end{aligned} \tag{27}$$

The consistency conditions (14) and (15) must be satisfied for $f_{ij}=q_{ij}$, $z_i=w_i$, $x_i=y_i$, $m=r=2$, $n=4$ and $a=2$. From (14) we obtain two relations, one for $i=j=1$ and another for $i=2$, $j=1$. We find that both relations are satisfied if

$$w_1^2+w_2^2=1. \tag{28}$$

Also since the expressions q_{1i}, $i=1,2$ are independent of q_{23} and q_{24} and similarly the expressions q_{2i}, $i=1,2$ are independent of q_{13} and q_{14}, as we see from (27), we must take $i=s=l$ in (15). Then we find that these equations are satisfied. Therefore we have to find the common solutions of (27) and (28). If we put

$$w_1=\sigma \qquad w_2=\pm(1-\sigma^2)^{1/2} \tag{29}$$

equations (27) reduce to the system of two equations

$$F_1\equiv y_1\partial_1\sigma+y_3\sigma\partial_3\sigma\mp y_3(1-\sigma^2)^{1/2}\partial_4\sigma-\sigma^2+1=0 \tag{30}$$

$$F_2\equiv y_1\partial_2\sigma\pm y_3(1-\sigma^2)^{1/2}\partial_3\sigma+y_3\sigma\partial_4\sigma\pm(1-\sigma)(1-\sigma^2)^{1/2}=0 \tag{31}$$

where $\partial_i\sigma=\partial\sigma/\partial y_i$, $i=1,2,\ldots,4$. The above expressions F_1 and F_2 satisfy the relation

$$[F_1,F_2]=0 \tag{32}$$

as expected, since (30) and (31) are consistent.

To solve (30) we find first the solution of the system

$$\frac{\mathrm{d}y_1}{y_1}=\frac{\mathrm{d}y_2}{0}=\frac{\mathrm{d}y_3}{y_3\sigma}=\frac{\mathrm{d}y_4}{\mp y_3(1-\sigma^2)^{1/2}}=\frac{\mathrm{d}\sigma}{\sigma^2-1}. \tag{33}$$

Solving this system we find that the general solution of (30) is given by

$$G_1\left(y_1\left(\frac{1+\sigma}{1-\sigma}\right)^{1/2},y_2,\frac{y_3}{(1-\sigma^2)^{1/2}},y_4\mp\frac{y_3\sigma}{(1-\sigma^2)^{1/2}}\right)=0 \tag{34}$$

where G_1 is an arbitrary function of its arguments. Also we find that the general solution of (31) is

$$G_2\left(y_1,y_2+y_1\left(\frac{1+\sigma}{1-\sigma}\right)^{1/2},\frac{y_3}{1-\sigma},y_4\mp y_3\left(\frac{1+\sigma}{1-\sigma}\right)^{1/2}\right)=0 \tag{35}$$

where again G_2 is an arbitrary function of its arguments. The common solutions of (30) and (31), i.e. the functions which are simultaneously of the form (34) and (35) are the following:

$$G\left(y_2+y_1\left(\frac{1+\sigma}{1-\sigma}\right)^{1/2},y_4\pm y_3\left(\frac{1+\sigma}{1-\sigma}\right)^{1/2},\frac{y_1y_3}{1-\sigma}\right)=0 \tag{36}$$

where G is an arbitrary function of its arguments, as in the previous cases. For example, if

$$G=\left[y_2+y_1\left(\frac{1+\sigma}{1-\sigma}\right)^{1/2}\right]\frac{1-\sigma}{y_1y_3}+c=0 \tag{37}$$

we obtain from (23), (25), (29) and the above relation

$$\gamma=\frac{y_3}{y_1(y_1-\mathrm{i}y_2)}[-\mathrm{i}(y_2+cy_1y_3)\pm(y_1^2-2cy_1y_2y_3-c^2y_1^2y_3^2)^{1/2}]. \tag{38}$$

The expression $(y_1/y_3)\gamma$ is a function not only of y_1 and y_2 but also of y_3.

If we write

$$g=w_1+\mathrm{i}w_2=\sigma\pm\mathrm{i}(1-\sigma^2)^{1/2}=1/\bar{g} \tag{39}$$

we obtain

$$\frac{1+\sigma}{1-\sigma}=-\left(\frac{g+1}{g-1}\right)^2 \qquad \frac{1}{1-\sigma}=-\frac{2g}{(g-1)^2}. \tag{40}$$

Then the solution G of (36) becomes a function of

$$A=-\mathrm{i}y_2+y_1\frac{g+1}{g-1} \tag{41a}$$

$$B=\mathrm{i}y_4\mp y_3\frac{g+1}{g-1} \tag{41b}$$

$$D=y_1y_3\frac{g}{(g-1)^2}. \tag{41c}$$

If we replace D by

$$D'=AB\pm 4D=(y_1-\mathrm{i}y_2)(\mp y_3+\mathrm{i}y_4)+\frac{2\mathrm{i}y_1y_4}{g-1}\pm\frac{2\mathrm{i}y_2y_3}{g-1} \tag{42}$$

and express A, B and D' in terms of E, $\bar{E}$, E' and $\bar{E}'$, choosing for B and D' the lower sign, (36) becomes

$$G\left(E+\frac{E+\bar{E}}{g-1},\ \bar{E}'+\frac{E'+\bar{E}'}{g-1},\ E\bar{E}'+\frac{E\bar{E}'-\bar{E}E'}{g-1}\right)=0. \tag{43}$$

We shall show now that the common solution of (30) and (31) cannot be obtained from an arbitrary function of four independent arguments. To show that we write the arbitrary functions from which the solutions of (30) and (31) are obtained in the form

$$\begin{aligned} &G_1'\left(y_2,\ y_2+y_1\left(\frac{1+\sigma}{1-\sigma}\right)^{1/2},\ y_4\pm y_3\left(\frac{1+\sigma}{1-\sigma}\right)^{1/2},\ \frac{y_1y_3}{1-\sigma}\right)=0\\ &G_2'\left(y_1,\ y_2+y_1\left(\frac{1+\sigma}{1-\sigma}\right)^{1/2},\ y_4\pm y_3\left(\frac{1+\sigma}{1-\sigma}\right)^{1/2},\ \frac{y_1y_3}{1-\sigma}\right)=0 \end{aligned} \tag{44}$$

respectively. The first arguments y_2 and y_1 of G_1' and G_2' cannot be replaced by a common expression, which is constructed from the four arguments of G_1' including y_2 and independently from the four arguments of G_2', including y_1. We can also say that if this was possible then (30) and (31), which are different equations, would have the same general solution. Of course this is impossible. This proves what we wanted to show.

From (3), (6), (22), (23), (25) and (39) we find that

$$\partial_a E'=\frac{E'+\bar{E}'}{E+\bar{E}}g\partial_a E \qquad \partial_a\bar{E}'=\frac{E'+\bar{E}'}{E+\bar{E}}\frac{1}{g}\partial_a\bar{E} \qquad a=\xi,\zeta. \tag{45}$$

Using these relations we obtain

$$\begin{aligned} \partial_a A&=\partial_a\left(E+\frac{E+\bar{E}}{g-1}\right)=-\frac{E+\bar{E}}{(g-1)^2}\left(g_a-\frac{g-1}{E+\bar{E}}(gE_a+\bar{E}_a)\right)\\ \partial_a B&=\partial_a\left(\bar{E}'+\frac{E'+\bar{E}'}{g-1}\right)=-\frac{E'+\bar{E}'}{(g-1)^2}\left(g_a-\frac{g-1}{E+\bar{E}}(gE_a+\bar{E}_a)\right)\\ \partial_a D'&=\partial_a\left(E\bar{E}'+\frac{E\bar{E}'-\bar{E}E'}{g-1}\right)=-\frac{E\bar{E}'-\bar{E}E'}{(g-1)^2}\left(g_a-\frac{g-1}{E+\bar{E}}(gE_a+\bar{E}_a)\right). \end{aligned} \tag{46}$$

From (43) and (46) we have

$$\partial_a G(A, B, D') = -\frac{1}{(g-1)^2}[(E+\bar{E})\partial_A G + (E'+\bar{E}')\partial_B G + (E\bar{E}' - \bar{E}E')\partial_{D'} G] \times \left(g_a - \frac{g-1}{E+\bar{E}}(gE_a + \bar{E}_a)\right) = 0 \tag{47}$$

which means that at least one of the relations

$$(E+\bar{E})\partial_A G + (E'+\bar{E}')\partial_B G + (E\bar{E}' - \bar{E}E')\partial_{D'} G = 0 \tag{48}$$

$$g_a = \frac{g-1}{E+\bar{E}}(gE_a + \bar{E}_a) \tag{49}$$

must be satisfied.

Dividing (48) by $g-1$ we obtain

$$(A-E)\partial_A G + (B-\bar{E}')\partial_B G + (D' - E\bar{E}')\partial_{D'} G = 0. \tag{50}$$

Suppose that this relation holds, which means that the g and therefore the σ we get from it does satisfy (30) and (31). Since, E, $\bar{E}'$ and $E\bar{E}'$ cannot be obtained from the arguments A, B and D' of (43), this can happen only if $G = G(A)$ or $G = G(B)$ or $G = G(D')$. Solving these relations for A, B and D', respectively, we get $A = -\mathrm{i}c_1$, $B = -\mathrm{i}c_2$, $D' = c_3$, from which we obtain

$$g = \frac{c_1 + \mathrm{i}\bar{E}}{c_1 - \mathrm{i}E} \qquad g = \frac{c_2 + \mathrm{i}E'}{c_2 - \mathrm{i}\bar{E}'} \qquad g = \frac{c_3 - \bar{E}E'}{c_3 - E\bar{E}'}. \tag{51}$$

Since we must have $g\bar{g} = 1$ the constants c_1, c_2 and c_3 must be real. However, if g is given by any one of the expressions (51) and E' and $\bar{E}'$ satisfy (45) we find that the function g obeys (49). Therefore even if (48) is satisfied g must obey (49), i.e. (49) is always satisfied. This means that we have found the Bäcklund transformations (45), where the pseudopotential g (Wahlquist and Estabrook 1975) can be obtained as a solution of (49). In this formalism the pseudopotential came out from the arbitrary function G.

The general solution of (49) is

$$g = \frac{c - \mathrm{i}\bar{E}}{c + \mathrm{i}E} \tag{52}$$

where c is an arbitrary real constant. Substituting this expression in (45) and solving the resulting equations we obtain

$$E' = \frac{E + \mathrm{i}c_1}{\mathrm{i}c_2 E + c_3} \tag{53}$$

where c_1, c_2 and c_3 are arbitrary real constants. This is Ehler's transformation (Ehler 1957).

The expression of (23) for γ includes the γ of (9) as a special case. We expect therefore the E' which is obtained from the γ of (21) to be related to the solution E we started from by an Ehler transformation. To show that let us write

$$g = 1 + cx_1x_2 + \mathrm{i}(-2cx_1x_2 - c^2x_1^2x_2^2)^{1/2} \tag{54}$$

for which $g\bar{g} = 1$. Then the Bäcklund transformations will be of the form of (45). But using these equations we find that the g of (54) satisfies (49). Therefore the E' of this case is again of the form of (53).

Another interesting feature of our formalism is the following. The g for which (43) and (45) are satisfied obey (49). Therefore modulo an arbitrary constant to each E there corresponds one and only one g. Equating the three expressions for g of (51) we obtain two equations which can be solved for E' and $\bar{E}'$ in terms of E and $\bar{E}$. If we do that we find again the Ehler transformation (53). In this approach we do not have to solve (49) nor the relation we get if we substitute the solution of (49) in (45).

Concluding we can say that we have presented a method by which we can derive Bäcklund transformations in a systematic way. By this method we have found for the ansatz of (23) the Ehler transformation, for every g derived from (43), i.e. in a case in which the functions appearing in the ansatz belong to a wide class of functions. Of course, by this method we can derive Bäcklund transformations not only for the Ernst equation but also for other equations.

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